SAINT IGNATIUS' COLLEGE

Trial Higher School Certificate

2007

MATHEMATICS EXTENSION 1

8:45am – 10:50am
Friday 31st August 2007

Directions to Students

• Reading Time: 5 minutes	• Total Marks: 84
• Working Time: 2 hours	
• Write using blue or black pen (sketches in pencil).	• Attempt Questions 1 – 7
• Board approved calculators may be used	• All questions are of equal value
• A table of standard integrals is provided at the back of this paper.	
• All necessary working should be shown in every question.	
• **Answer each question in the booklets provided and clearly label your name and teacher's name.**	

QUESTION 1 (use a SEPARATE writing booklet)

(a) Given that if $y = \sin^{-1}\left(\frac{x}{a}\right)$ then $\frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$, where $x < |a|$,

write down an expression for $\frac{dy}{dx}$ if y equals:

(i) $\sin^{-1}(x)$

(ii) $\sin^{-1}\left(\frac{x}{7}\right)$

(iii) $\sin^{-1}(7x)$ (3M)

(b)

(i) If $f(x) = x^3 - 3x^2 - 4x + 12$, show that $f(3) = 0$ (1M)

(ii) Hence solve $x^3 - 3x^2 - 4x + 12 = 0$ (2M)

(c) Prove that

$$\frac{1}{3}\left[n(n+1)(n+2)\right] + (n+1)(n+2) = \frac{1}{3}(n+1)(n+2)(n+3)$$

(2M)

(d) Solve for x

$$\frac{x-2}{x+4} \geq \frac{1}{3}$$

(4M)

QUESTION 2 (use a SEPARATE writing booklet)

(a) A particle P moves in a straight line so that its distance x from a central fixed point O at time t is given by

$$x = 2\sin\left(5t + \frac{\pi}{6}\right)$$

(i) Write down an expression for the velocity $\dot{x}$

(ii) Write down an expression for the acceleration $\ddot{x}$

(iii) The particle P is said to be executing Simple Harmonic Motion. Explain why this is so.

(3M)

(b) The parametric equations of a parabola are

$x = 2t$

$y = 2t^2$

Find the equation of the tangent to this parabola at the point

$P(2,2)$ (2M)

(c) Given that $x^4 + 3x^2 - 100 = 0$ has a root near $x = 3$, use Newton's method once to find a better approximation, giving your answer in exact form.

(2M)

(d) Find the numerical value of the co-efficient of x^0 in the expansion of

$\left(2x^2 - \frac{1}{2x}\right)^{12}$ (3M)

(e) Find the sum of the six co-efficients (including the co-efficient of x^0) in the expansion of $(3-x)^5$

(2M)

QUESTION 3 (use a SEPARATE writing booklet)

(a) Using the table of standard integrals, evaluate $\int_0^2 \frac{8dx}{x^2+4}$ (2M)

(b) Find in radians all the values of x in the domain $0 \le x \le \pi$ which satisfy the equation

$$\sin 2x - \sin x = 0$$ (3M)

(c) If α, β and γ are the roots of the cubic equation

$$2x^3 - 8x^2 + x + 12 = 0,$$

write down the value of

(i) $\alpha + \beta + \gamma$

(ii) $\alpha\beta + \beta\gamma + \gamma\alpha$

(iii) $\alpha^2 + \beta^2 + \gamma^2$ (3M)

(d)

(i) Show that $\frac{d}{dx}\left(y^2\right) = 2y\frac{dy}{dx}$

(ii) Deduce from (i) that if $x^2 + 4y^2 = 5$ then $\frac{dy}{dx} = \frac{-x}{4y}$

(iii) A particle moves at a constant speed of k ms^{-1} on the circumference of the curve

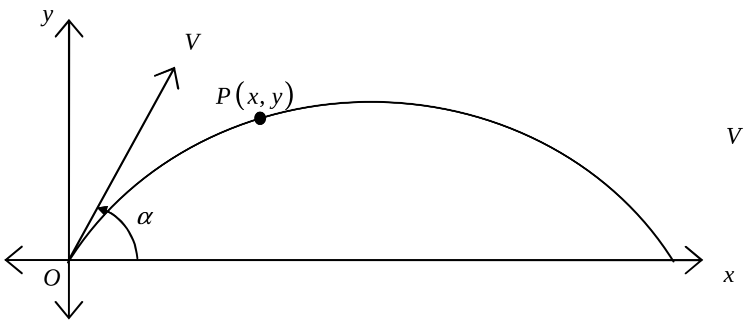
$$x^2 + 4y^2 = 5$$

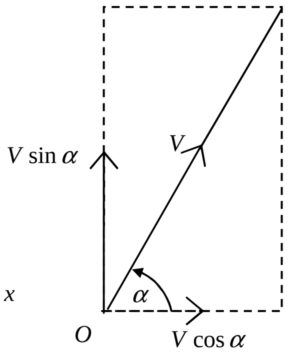
Find $\frac{dy}{dt}$ at the point P where $x = 2$, $y = -\frac{1}{2}$ and $\frac{dx}{dt} = 2$ ms^{-1}

(iv) What is the exact value of k, the constant speed?

(4M)

QUESTION 4 (use a SEPARATE writing booklet)





(The above diagrams may help in your presentation)

A particle is projected with a velocity V from a point O at an angle of α to the horizontal, in the x, y plane. In the usual notation using the calculus or otherwise, prove that the horizontal distance x and the vertical distance y, travelled by the particle at time t are given by:

(i) $x = (V\cos\alpha)t$ (2M)

(ii) $y = (V\sin\alpha)t - \frac{1}{2}gt^2$
where g is the acceleration due to gravity. (3M)

(iii) By combining (i) and (ii) deduce that:

$$y = x\tan\alpha - \frac{gx^2}{2V^2}\left(1+\tan^2\alpha\right)$$ (3M)

(iv) Any of the above results may be used in attempting this problem:

A football is kicked at $15ms^{-1}$ and just passes over a crossbar 5m high and 15m away.

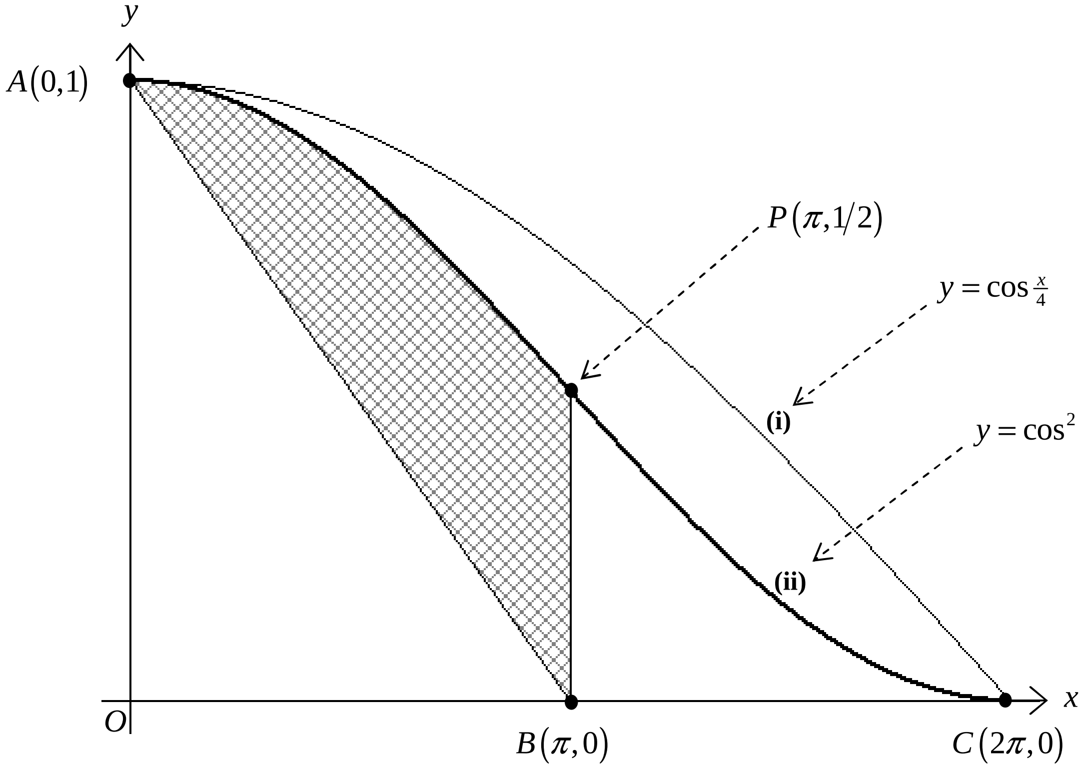
Taking $g = 10ms^{-2}$ vertically downwards, show that if α is the angle of projection then $\alpha = \frac{\pi}{4}$ or $\alpha = \tan^{-1} 2$

(3M)

(v) If the two values of α (from part iv) are given by A and B (respectively) where $A > B$, write down the exact value of $\tan(A-B)$

(1M)

QUESTION 5 (use a SEPARATE writing booklet)



The above diagram shows two curves

$y = \cos\frac{x}{4}$ (i) (thin line)

$y = \cos^2\left(\frac{x}{4}\right)$ (ii) (thick line)

Also, A is the point $(0,1)$, B is the point $(\pi,0)$ and C is the point $(2\pi,0)$

(i) Prove that the area enclosed by the curve $y = \cos\dfrac{x}{4}$ and the x and y axes is 4 square units.

(2M)

(ii) The shaded region $APBA$ is enclosed by $y = \cos^2\left(\dfrac{x}{4}\right)$ and the lines AB and $x = \pi$. Prove that the area of this shaded region is 1 square unit.

(3M)

(iii) Prove that $\cos^4\left(\dfrac{x}{4}\right) = \dfrac{1}{8}\left(3 + 4\cos\dfrac{x}{2} + \cos x\right)$

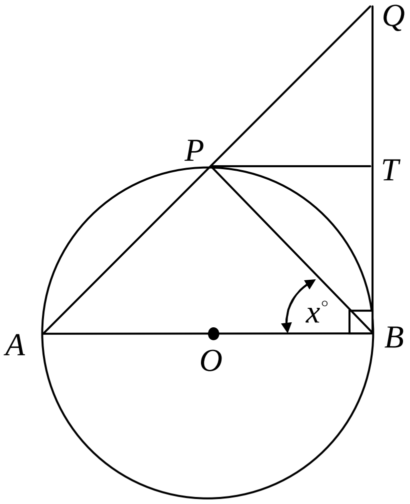
(3M)

(iv) The shaded region is now rotated through 2π radians about the x-axis. Show, using the result of part (iii) or otherwise that the exact volume of the solid generated is $\pi\left(1 + \dfrac{\pi}{24}\right)$ cubic units.

(4M)

QUESTION 6 (use a SEPARATE writing booklet)

(a) If $y = 2007^x$, find an expression for $\frac{dy}{dx}$ in terms of x (2M)



(b) AB is the diameter of a circle, centre O. P is a point on the circumference. The tangents at B and P meet at T, and AP, BT are produced to meet at Q, and $A\hat{B}P = x^\circ$

(i) Copy the diagram into your script and explain why $A\hat{P}B = 90^\circ$

(ii) Noting without any explanation, that $O\hat{B}Q = 90^\circ$ or otherwise, explain why $T\hat{Q}P = x^\circ$

(iii) Explain why $T\hat{P}Q = x^\circ$

(iv) If $x = 30$ show $PQ = 3AP$ (4M)

(c)

(i) In the usual notation, prove that $\ddot{x} = v\frac{dv}{dx}$ (1M)

(ii) A particle moves in a straight line and its acceleration at any time t is given by $\ddot{x} = v\frac{dv}{dx} = -e^{-2x}$, where x is the displacement and v the velocity at time t. Also, when $x = 0$, $v = 1$.

By starting with the result of part (i) or otherwise, prove that $v = e^{-x}$.
(3M)

(iii) It is also known that when $t = 0$, $x = 0$. Deduce from (ii), or otherwise prove, that the displacement x at time t is given by $x = \ln(t+1)$
(2M)

QUESTION 7 (use a SEPARATE writing booklet)

(a) If $y = 3\cos^{-1}\left(\dfrac{x}{2}\right)$, write down (i) the domain of x

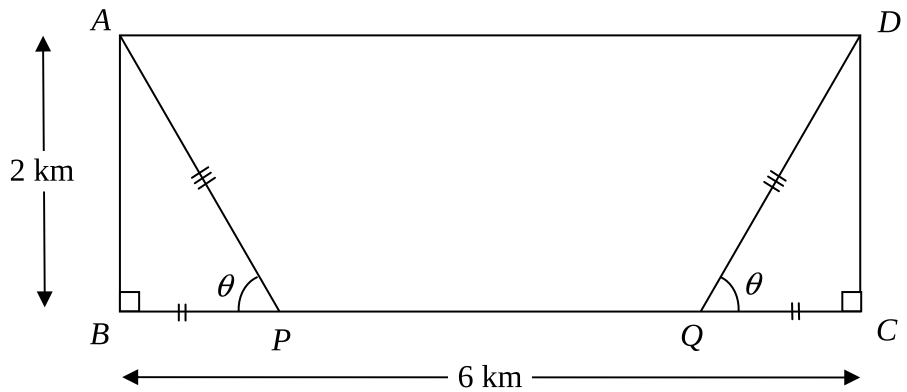
(ii) the range of y (2M)

(b) If $y = f(x) = \dfrac{kx+l}{mx-k}$ (where k, l, m are constants), prove that $x = f(y)$

(2M)

(c) In attempting the problem below, you may assume without proof that:

- If $y = \cot\theta$, $\dfrac{dy}{d\theta} = -\text{cosec}^2\,\theta$
- If $y = \text{cosec}\,\theta$, $\dfrac{dy}{d\theta} = -\text{cosec}\,\theta\cot\theta$



The diagram shows a house at A, a school at D and a straight canal BC, where $ABCD$ is a rectangle with $AB = 2km$ and $BC = 6km$.

During the winter, when the canal freezes over, Danny travels from A to D by walking to a point P on the canal, skating along the canal to a point Q and then walking from Q to D. The points P and Q are chosen so that the angles APB and DQC are both equal to θ, $AP = QD$, $BP = QC$.

(i) Show that $PQ = (6 - 4\cot\theta)$ (1M)

(ii) Given that Danny walks at a constant speed of $4kmh^{-1}$, and skates at a constant speed of $8kmh^{-1}$ show that the time, T minutes, taken for Danny to go from A to D along this route is given by

$$T = 15(3 + 4\,\text{cosec}\,\theta - 2\cot\theta)$$ (3M)

(iii) Show clearly and carefully that, as θ varies, Danny's minimum time for the journey is approximately 97 minutes.

(4M)

END OF PAPER

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax\ \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\quad x > 0$

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MATHEMATICS EXTENSION 1
Solutions, Marking Scheme & Comments

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QUESTION 1

Suggested Solutions	Max Mark	Your Mark	Marker's Comments
(a) (i) $\frac{1}{\sqrt{1-x^2}}$ $x<\lvert 1\rvert$ ✓ (ii) $\frac{1}{\sqrt{49-x^2}}$ $x<\lvert 7\rvert$ ✓	2		
(iii) $\frac{1}{\sqrt{\left(\frac{1}{7}\right)^2-x^2}}$ OR $\frac{7}{\sqrt{1-49x^2}}$ $x<\left\lvert\frac{1}{7}\right\rvert$ ✓ No penalty for omitting domain of x	1		
(b) (i) Here $f(3)=27-27-12+12=0$	1	✓	
(ii) From (i) $f(x)=(x-3)(x^2-4)$ Here $f(x)=(x-3)(x-2)(x+2)=0$	1	✓	
Hence the required solutions are $x=3,2,-2$ ✱	1	✓	✱ No mark awarded if soln. not written.
(c) L.H.S=$\frac{1}{3}[n(n+1)(n+2)+3(n+1)(n+2)]$ We take out the factors $[(n+1)(n+2)]$	1	✓	Some tried Induction.
To get L.H.S$=\frac{1}{3}[(n+1)(n+2)](n+3)$ (Award 2 marks for L.H.S = R.H.S by direct multiplication) $\frac{1}{3}(n^3+6n^2+11n+6)$	1	✓	(+1) for 1st two steps, if no solution was shown.
(d) $\frac{x-2}{x+4}$ is undefined if $x+4=0$ ie if $x=-4$ Multiply both sides by $3(x+4)^2$ To get $3(x-2)(x+4)\geq(x+4)^2$, $(x\neq-4)$	1 ✓		if $x\neq-4$ had to be mentioned, otherwise −1.
ie $3x^2+6x-24\geq x^2+8x+16$, $(x\neq-4)$	1 ✓		
Hence $x^2-x-20\geq0$, $(x\neq-4)$ ie $(x+4)(x-5)\geq0$, but $x\neq-4$	1 ✓		
$\therefore x<-4$ *or* $x\geq5$ To gain the 4[th] mark, the solution needs to mention $x\neq-4$ just the once.	1 ✓		

QUESTION 2

Suggested Solutions	Max Mark	Your Mark	Marker's Comments
(a) (i) $\dot{x} = 10\cos\left(5t + \frac{\pi}{6}\right)$ ✓	1		
(ii) $\ddot{x} = -50\sin\left(5t + \frac{\pi}{6}\right)$ ✓	1		
(iii) $\ddot{x} = -5^2\left(2\sin\left(5t + \frac{\pi}{6}\right)\right) = -5^2 x \quad \otimes$ At least 2 explanations are acceptable Eg (α) $x = 2\sin\left(5t + \frac{\pi}{6}\right) = 2\cos\left(5t - \frac{\pi}{3}\right)$, defines S.H.M (β) Equation $\otimes$ $\ddot{x} = -n^2 x \quad (n = 5)$ defines S.H.M ✓	1		most students used $\ddot{x} = -n^2 x$
(b) Here $\frac{dy}{dx} = \frac{dy}{dt} \bullet \frac{dt}{dx} = 4t \times \frac{1}{2} = 2t$ ✓ OR $\frac{dy}{dt} = 2$ At P, $t = 1$ $\therefore$ Equation is $y = 2x - 2$ OR $2x - y - 2 = 0$ ✓	1 1		
(c) Here $f(x) = x^4 + 3x^2 - 100 \quad \therefore f(3) = 8$ $f'(x) = 4x^3 + 6x \quad f'(3) = 126$ } both correct ✓ In the usual notation $x_1 = x_0 - \frac{f(x)}{f'(x)} = 3 - \frac{8}{126}$ $= 2\frac{59}{63}$ $\left(= \frac{185}{63}\right)$ ✓	1 1		
(d) (i) Here $T_{R+1} = \binom{12}{R}\left(\frac{-x^{-1}}{2}\right)^R \bullet (2x^2)^{12-R}$ ✓ Power of x in $T_{R+1} = -R + 24 - 2R$ $= 24 - 3R = 0$ when $R = 8$ ✓ So $T_9 = \binom{12}{8}\left(\frac{1}{2^8}\right)2^4 = \frac{12.11.10.9}{4.3.2.1} \times \frac{1}{16}$ $= \frac{495}{16}$ $\left(= 30.9353 = 30\frac{15}{16}\right)$ ✓	1 1 1 1		
(ii) To sum the 6 co-efficients, set $x = 1$ ✓ Required sum $(3 - 1)^5 = 2^5 = 32$ OR $3^5 + \binom{5}{1}3^4(-1) + \binom{5}{2}3^3(-1)^2 + \binom{5}{3}3^2(-1)^3 + \binom{5}{4}3(-1)^4 + (-1)^5$ ✓	1 1		

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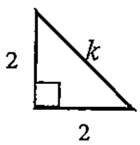
QUESTION 3

Suggested Solutions	Max Mark	Your Mark	Marker's Comments
(a) $I=\left[4\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$	1	✓	
$=4\left[\tan^{-1}1-\tan^{-1}0\right]=\frac{4\pi}{4}=\pi$	1	π ✓	
(b) $2\sin x\cos x-\sin x=0$	1	✓	
$\therefore \sin x(2\cos x-1)=0\rightarrow \sin x=0$ or $\cos x=\frac{1}{2}$	1	✓	
So $x=0,\pi$ or 2π or $x=\frac{\pi}{3}$ or $\frac{5\pi}{3}$			
But $0\le x\le\pi$ $x=0,\frac{\pi}{3},\pi$ (Penalty of 1 if degrees are used instead of radians)	1	$0, \frac{\pi}{3}, \pi$ ✓	Any answers other than these three did not get a mark here.
(c) (i) 4	1	4 ✓	
(ii) $\frac{1}{2}$	1	$\frac{1}{2}$ ✓	
(iii) $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\alpha\gamma)=15$	1	15 ✓	
(d) (i) Use product rule $\frac{d}{dx}(y.y)=yy'+y'y=2yy'$ OR note $\frac{d}{dx}(y^2)=\frac{d}{dy}(y^2)\bullet\frac{dy}{dx}=2yy'$	1	✓ $2y\frac{dy}{dx}$	Some tried to reproduce the Question – No marks. All other methods were accepted.
(ii) Differentiate both sides with respect to x To get $2x+8y\bullet\frac{dy}{dx}=0$ $\therefore \frac{dy}{dx}=\frac{-x}{4y}$	1	$\frac{-x}{4y}$ ✓	
(iii) $\frac{dy}{dt}=\frac{dy}{dx}\bullet\frac{dx}{dt}=\frac{-x}{4y}(2)=2ms^{-1}$	1	2 ✓	
(iv) Note: we focus on <u>speed</u> (and not velocity) At P, speed along x axis = speed along the y axis $=2ms^{-1}$ $\therefore$ Resulting speed $=\sqrt{2^2+2^2}$ $k=2\sqrt{2}$	1	$2\sqrt{2}$ ✓	



QUESTION 4

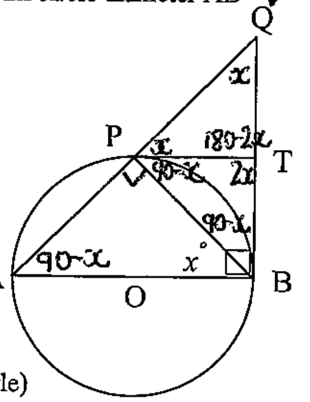
Suggested Solutions	Max Mark	Your Mark	Marker's Comments
(i) **HORIZONTAL MOTION** The initial velocity in a horizontal direction is V $\cos\alpha$ and since no forces are acting in the horizontal direction, this velocity will remain uniform. $\ddot{x}=0$ That is $\frac{dx}{dt}=V\cos\alpha$ ✓ Thus $x=(V\cos\alpha)t+c$ But $x=0$ when $t=0$, thus $c=0$ Hence $x=(V\cos\alpha)t$ ….(i) ✓	1 1		
(ii) **VERTICAL MOTION** The initial velocity in a vertical direction is $V\sin\alpha$. If we take the upwards direction as positive the particle experiences a uniform acceleration of –g due to gravity. Thus $\frac{d^2y}{dt^2}=-g$ ✓ Hence $\frac{dy}{dt}=-gt+c$ But $\frac{dy}{dt}=V\sin\alpha$ when $t=0$, thus $c=V\sin\alpha$ Thus $\frac{dy}{dt}=V\sin\alpha-gt$ ✓ Hence $y=(V\sin\alpha)t-\frac{1}{2}gt^2+k$ But $y=0$ when $t=0$, thus $k=0$ Giving $y=(V\sin\alpha)t-\frac{1}{2}gt^2$ ….(ii) ✓	1 1 1		Note: You must show the initial conditions to work out the constants.
(iii) Eliminating t From (i) $t=\frac{x}{V\cos\alpha}$ ✓ Substituting in (ii) $y=\frac{(V\sin\alpha)x}{V\cos\alpha}-\frac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2$ ✓ i.e $y=x\tan\alpha-\frac{gx^2}{2V^2}\sec^2\alpha=x\tan\alpha-\frac{gx^2}{2V^2}(1+\tan^2\alpha)$…(iii) ✓	1 1 1		Remember $\sec^2\alpha=1+\tan^2\alpha$
(iv) Using equation (iii) $5=15\tan\alpha-5-5\tan^2\alpha$, ✓ $5=15t-5-5t^2$ where $t=\tan\alpha$ $\therefore 5t^2-15t+10=0$ $t^2-3t+2=0$ ✓ $(t-2)(t-1)=0$ Hence $t=1$ and $t=2$ $\therefore \alpha=\frac{\pi}{4}$ & $\alpha=\tan^{-1}2$ ✓	2 1		1st mark: correct substitution 2nd mark: correct equation
(v) $\tan(A-B)=\frac{TanA-TanB}{1+TanATanB}=\frac{2-1}{1+2}=\frac{1}{3}$	1		N.B question A>B ∴ answer $=\frac{1}{3}$

MATHEMATICS EXTENSION 1
Solutions, Marking Scheme & Comments

QUESTION 5

Suggested Solutions	Max Mark	Your Mark	Marker's Comments
(i) From diagram, Area = $\int_0^{2\pi} \cos\frac{x}{4}\,dx$	1	✓	
i.e. A = $\left[4\sin\frac{x}{4}\right]_0^{2\pi} = 4\sin\frac{\pi}{2} - 0 = 4$	1	✓	
(ii) Shaded Area = $\int_0^{\pi} \cos^2\left(\frac{x}{4}\right)dx - \Delta BOA$ ✓	1		
$= \frac{1}{2}\int_0^{\pi}\left(\cos\frac{x}{2}+1\right)dx - \frac{1}{2}(\pi\times 1)$	$\cos^2 A = \cos 2A + 1$		
$= \frac{1}{2}\left[2\sin\left(\frac{x}{2}\right)+x\right]_0^{\pi} - \frac{\pi}{2}$ ✓	1	$\frac{1}{2}\left[2\sin\left(\frac{x}{2}\right)+x\right]_0^{\pi} - \frac{\pi}{2}$ $= \frac{1}{2}\left[2\times\sin\frac{\pi}{2}+\pi-\right] - \frac{\pi}{2}$	
$= \left[\sin\frac{\pi}{2}+\frac{\pi}{2}-0-0\right] - \frac{\pi}{2} = 1$ ✓	1	$= \sin\frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{2}$ $= \sin\frac{\pi}{2} = 1$	
(iii) Since $\cos^2\left(\frac{x}{4}\right) = \frac{1}{2}\left[1+\cos\left(\frac{x}{2}\right)\right]$	1	✓	
Therefore $\cos^4\frac{x}{4} = \frac{1}{4}\left[1+\cos\left(\frac{x}{2}\right)\right]^2$			
$\therefore \cos^4\frac{x}{4} = \frac{1}{4}\left[\left(1+2\cos\frac{x}{2}+\cos^2\left(\frac{x}{2}\right)\right)\right]$	1	✓	
$= \frac{1}{4}\left[1+2\cos\left(\frac{x}{2}\right)+\frac{1}{2}(1+\cos x)\right]$			
$= \frac{1}{8}\left(3+4\cos\frac{x}{2}+\cos x\right)$	1	✓	
(iv) Required Volume $= \pi\int_0^{\pi} \cos^4\left(\frac{x}{4}\right)dx - v_c$	1	✓	
where v_c = volume of a cone with radius unity and height π.			
ie $v_c = \frac{1}{3}\cdot\pi\cdot 1^2\cdot\pi = \frac{\pi^2}{3}$	1 ✓		
$\therefore$ Required Volume $= \frac{\pi}{8}\int_0^{\pi}\left(3+4\cos\frac{x}{2}+\cos x\right)dx - \frac{\pi^2}{3}$ from ~~(iii)~~			
$= \frac{\pi}{8}\left[3x+8\sin\frac{x}{2}+\sin x\right]_0^{\pi} - \frac{\pi^2}{3}$	1 ✓		
$= \frac{\pi}{8}[3\pi+8+0] - \frac{\pi}{8}[0+0+0] - \frac{\pi^2}{3}$	→	$\frac{3\pi^2}{8}+1 \quad -\frac{\pi^2}{3}$	
$= \frac{8\pi}{8} + \pi^2\left[\frac{3}{8}-\frac{1}{3}\right] = \pi + \frac{\pi^2}{24} = \pi\left[1+\frac{\pi}{24}\right]$	1 ✓	Few got to the end. Yet if 4 right steps full mark was awarded.	

QUESTION 6

Suggested Solutions	Max Mark	Your Mark	Marker's Comments
(a) Taking logs to base e of each side $(\ln y)=(x)\ln 2007$ Now ✓	1		poorly answered some Extension 2 students O.K.
differentiate both sides with respect to x: $\frac{1}{y}\cdot\frac{dy}{dx}=\ln 2007$			
now multiply both sides by $y=(2007)^x$ to get $\frac{dy}{dx}=(2007)^x(\ln 2007)$ ✓	1		
(b) (i) $A\hat{P}B=90^\circ$ since it is the angle in the semi circle diameter AB ✓	1		Note: You cannot assume PT ∥ AB
(ii) $O\hat{B}Q=90^\circ$ (given) $\therefore P\hat{B}Q=90^\circ-x^\circ$ But $A\hat{P}B=P\hat{B}Q+P\hat{Q}B$ (Ext $\angle$ of Δ = sum of interior opp $\angle$'s) $90^\circ=90^\circ-x^\circ+P\hat{Q}B$ $\therefore P\hat{Q}B=x^\circ=T\hat{Q}P$ (same angle) 	1		I have written in the correct values for each angle in terms of the pronumerals.
(iii) TP=TB (tangents from ext point to a circle are equal) $T\hat{P}B=T\hat{B}P=90^\circ-x^\circ$ So $T\hat{P}Q=[90^\circ-(90^\circ-x^\circ)]=x^\circ$	1		
(iv) $\frac{AP}{AQ}=\frac{AB\cos 60^\circ}{\left(\frac{AB}{\cos 60^\circ}\right)}=\left(\frac{1}{2}\times\frac{1}{2}\right)=\frac{1}{4}$ $\therefore PQ=3AP$	1		
(c) (i) $\ddot{x}=\frac{dv}{dt}=\frac{dx}{dt}\cdot\frac{dv}{dx}=v\frac{dv}{dx}$	1		
(ii) Using part (i) $v\frac{dv}{dx}=-e^{-2x}$ and rearranging, $\int v\,dv=-\int e^{-2x}dx$	1		* Some used $\frac{d}{dx}\left(\frac{1}{2}v^2\right)=\ddot{x}$ this was O.K.
Thus $\frac{v^2}{2}=\frac{1}{2}e^{-2x}+K$ Set $x=0$ and $v=1$	1		
To get $\frac{1}{2}=\frac{1}{2}(1)+K\Rightarrow K=0$ thus $v^2=e^{-2x}$ so $v=e^{-x}$	1		
(iii) From part (ii) $\frac{dx}{dt}=e^{-x}$ So again rearranging, $\frac{dx}{e^{-x}}=dt$ and hence $\int e^x dx=\int dt\Rightarrow e^x=t+c$	1		Note: $\frac{dx}{dt}=e^{-x}$ to assist in correct integration
set $t=0$ and $x=0$, (data), $e^0=0+c$ $\therefore c=1$ ie $e^x=t+1$ Finally take logs of each side to base e, to get $x=\ln\lvert t+1\rvert$	1		

QUESTION 7

Suggested Solutions	Max Mark	Your Mark	Marker's Comments
$-2 \le x \le 2$ (a) (i) $x \le \|2\|$ (OR) (ii) $0 \le y \le 3\pi$	1+1		so many students didn't get these correct. disappointing that
(b) Note that $mxy - ky = kx + l \quad \therefore mxy - kx = ky + l$ and so $x(my-k) = ky + l \Rightarrow x = \dfrac{ky+l}{my-k} = f(y)$ (neither x, or $y = \dfrac{k}{m}$)	1 1		→ attempting to make x the subject → correct result
(c) (i) $PQ = BC - 2BP$ and in right angled ΔABP Tan $\theta = \dfrac{2}{BP} \quad \therefore \cot\theta = \dfrac{BP}{2} \quad \therefore BP = 2\cot\theta$ Hence $PQ = BC - 2(2\cot\theta) = 6 - 4\cot\theta$	1		
(ii) By similar reasoning as in (i) $AP = 2\operatorname{cosec}\theta$ so $AP + QD = 4\operatorname{cosec}\theta$ Now focus on time in MINUTES (t_m) t_m for D to travel $AP + QD = 60\left(\dfrac{4\operatorname{cosec}\theta}{4}\right) = 60\operatorname{cosec}\theta$ t_m for D to travel $PQ = 60\dfrac{(6-4\cot\theta)}{8} = (45 - 30\cot\theta)$ Let T=Total t_m, so $T = 60\operatorname{cosec}\theta + 45 - 30\cot\theta$ $= 15(3 + 4\operatorname{cosec}\theta - 2\cot\theta)$	1 1 1		→ correct expression for length of AP or QD. → correct expression for time taken (even if in hours) → correct conversion from time in hours to time in minutes.
(iii) $\therefore \dfrac{dT}{d\theta} = -60\operatorname{cosec}\theta\cot\theta + 30\operatorname{cosec}^2\theta$ $= 30\operatorname{cosec}^2\theta(-2\cos\theta + 1)$ Note $30\operatorname{cosec}^2\theta > 0$ So (in discussing) the sign of $\dfrac{dT}{d\theta}$ we need focus on sign of $(-2\cos\theta + 1)$ Now for a Max or a Min $\dfrac{dT}{d\theta} = 0$ so $-2\cos\theta + 1 = 0$ or $\cos\theta = \dfrac{1}{2}$ and $\theta = \dfrac{\pi}{3}$ (Note θ is acute) If $\theta = \dfrac{\pi}{3}$, $T = 15\left(\dfrac{8}{\sqrt{3}} + 3 - \dfrac{2}{\sqrt{3}}\right) = 45 + 30\sqrt{3}$ $\approx 45 + 51.963 \approx 96.963\ldots$ Hence for $\theta = \dfrac{\pi}{3}$, T is just slightly below 97 minutes. We have now to show this is a minimum. Note $\dfrac{dT}{d\theta}$ takes the same sign as $(1 - 2\cos\theta)$ Now if $0 \le \theta < \dfrac{\pi}{3}$, $\cos\theta > \dfrac{1}{2} \quad \therefore 1 - 2\cos\theta < 0$ Now if $\dfrac{\pi}{3} < \theta \le \dfrac{\pi}{2}$, $\therefore \cos\theta < \dfrac{1}{2}$, $\therefore 1 - 2\cos\theta > 0$ Hence as θ passes through $\dfrac{\pi}{3}$, $\dfrac{dT}{d\theta}$ goes from negative through 0 to positive. Hence at $\theta = \dfrac{\pi}{3}$ we get the least value of T. when $\theta = \dfrac{\pi}{3}$, $T \doteqdot 97$ minutes	1 1 1 1		→ correct expression for $\dfrac{dT}{d\theta}$ * many students didn't use the given results at beginning of question. → correct solution for θ $\left(\dfrac{\pi}{3}\right)$ → testing that $\theta = \dfrac{\pi}{3}$ gives a minimum time → substituting $\theta = \dfrac{\pi}{3}$ into expression for T and obtaining $T \doteqdot 97$ min.

QUESTION 7 (c) (iii)			
Suggested Solutions	**Max Mark**	**Your Mark**	**Marker's Comments**
2[nd] way of showing that $\theta = \frac{\pi}{3}$ gives a minimum time for Danny's journey.			
Note that $\frac{1}{30}\frac{dT}{d\theta} = \operatorname{cosec}^2\theta(1 - 2\cos\theta)$	1		
$\therefore \frac{1}{30}\frac{d^2T}{d\theta^2} = (1 - 2\cos\theta)\left[-2\operatorname{cosec}^2\theta\cot\theta\right] + \operatorname{cosec}^2\theta(2\sin\theta)\ldots(\text{L})$			
RHS of L $= \left[-2\operatorname{cosec}^2\theta\cot\theta + 4\operatorname{cosec}^2\theta\cot\theta\cos\theta\right] + 2\operatorname{cosec}\theta$ $= 2\operatorname{cosec}^2\theta\cot\theta(2\cos\theta - 1) + 2\operatorname{cosec}\theta$ $= 2\operatorname{cosec}^2\frac{\pi}{3}\cot\frac{\pi}{3}(2\cos\frac{\pi}{3} - 1) + 2\operatorname{cosec}\frac{\pi}{3}, \text{ if } \theta = \frac{\pi}{3}.$ $= 0 + \frac{4}{\sqrt{3}}, \text{ as } 2\cos\frac{\pi}{3} = 1$ $= \frac{4}{\sqrt{3}}$	1		
$\therefore \frac{1}{30}\cdot\frac{d^2T}{d\theta^2} = \frac{4}{\sqrt{3}}$ $\therefore \frac{d^2T}{d\theta^2} = 30 \times \frac{4}{\sqrt{3}} = 40\sqrt{3} > 0$	1		
$\therefore \theta = \frac{\pi}{3}$ gives a minimum time for Danny's journey.			
Source of question 7(c) University of London GCE Pure Mathematics 405, January 1988.	1		
371/405/420 Question 14			